

RANDOM REAL TREES

by

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Abstract

We survey recent developments about random real trees, whose prototype is the Continuum Random Tree (CRT) introduced by Aldous in 1991. We briefly explain the formalism of real trees, which yields a neat presentation of the theory and in particular of the relations between discrete Galton-Watson trees and continuous random trees. We then discuss the particular class of self-similar random real trees called stable trees, which generalize the CRT. We review several important results concerning stable trees, including their branching property, which is analogous to the well-known property of Galton-Watson trees, and the calculation of their fractal dimension. We then consider spatial trees, which combine the genealogical structure of a real tree with spatial displacements, and we explain their connections with superprocesses. In the last section, we deal with a particular conditioning problem for spatial trees, which is closely related to asymptotics for random planar quadrangulations.

Résumé

Nous discutons certains développements récents de la théorie des arbres réels aléatoires, dont le prototype est le CRT introduit par Aldous en 1991. Nous introduisons le formalisme d'arbre réel, qui fournit une présentation élégante de la théorie, et en particulier des relations entre les arbres de Galton-Watson discrets et les arbres continus aléatoires. Nous discutons ensuite la classe des arbres auto-similaires appelés arbres stables, qui généralisent le CRT. Nous présentons plusieurs résultats importants au sujet des arbres stables, notamment leur propriété de branchement, analogue continu d'une propriété bien connue pour les arbres de Galton-Watson, et le calcul de leurs dimensions fractales. Nous considérons ensuite les arbres spatiaux, qui combinent la structure généalogique d'un arbre réel avec des déplacements dans l'espace, et nous expliquons leurs liens avec les superprocessus. Dans la dernière partie, nous traitons un conditionnement particulier des arbres spatiaux, qui est étroitement lié à certains résultats asymptotiques pour les quadrangulations planes aléatoires.

Introduction

The purpose of this paper is to give an overview of recent work about continuous genealogical structures and their applications. The interest for these continuous branching structures first arose from their connections with the measure-valued branching processes called superprocesses, which have been studied extensively since the end of the eighties. Since superprocesses are obtained as weak limits of branching particle systems, it is not surprising that their evolution should be coded by a kind of continuous genealogical structure, and Perkins used non-standard analysis to give a precise definition of this structure (see in particular [56]). A little later, the Brownian snake construction of (finite variance) superprocesses [42],[43] provided another way of describing the underlying genealogy. This construction made it clear that the genealogy of superprocesses, or equivalently of Feller's branching diffusion process (which corresponds to the total mass of a superprocess), could be coded by the structure of excursions of linear Brownian motion above positive levels. The Brownian snake approach had significant applications to sample path properties of superprocesses [49] or to their connections with partial differential equations [44], [45].

In a series of papers [1], [2], [3] at the beginning of the nineties, Aldous developed the theory of the Continuum Random Tree, and showed that this object is the limit as $n \rightarrow \infty$, in a suitable sense, of rescaled critical Galton-Watson trees conditioned to have n vertices (see Theorem 3.1 below). Although the CRT was first defined as a particular random subset of the space ℓ^1 , it was identified in [3] as the tree coded by a normalized Brownian excursion, in a way very similar to the Brownian snake approach to superprocesses (note however that the CRT is related to a Brownian excursion normalized to have duration 1, whereas in the Brownian snake approach it is more natural to deal with unnormalized excursions).

In a subsequent paper, Aldous [4] suggested the definition of the so-called integrated super-Brownian excursion (ISE), which combines the genealogical structure of the CRT with spatial Brownian displacements. A simple way of looking at ISE is to view it as the uniform measure on the range of a Brownian snake driven by a normalized Brownian excursion (see Section IV.6 in [45], and also Definition 6.1 below). ISE turned out to appear in asymptotics for several models of statistical mechanics: See in particular [17] and [34]. In certain cases however, the continuous branching structure of ISE (or equivalently of the CRT) is not appropriate to describe the asymptotics of the model: For instance, the results of [35] for oriented percolation involve the canonical measure of super-Brownian motion, whose genealogical structure is described by a Brownian excursion with height greater than 1, rather than a normalized excursion. This suggests that rather than concentrating on the normalized excursion it is worthwhile to deal with various types of Brownian excursions which correspond to different conditionings of the fundamental object which is the Itô measure.

Both the Brownian snake approach to superprocesses and Aldous' representation of the CRT correspond to the fact that the genealogical structure of Feller's branching diffusion process can be coded by positive Brownian excursions.

sions, in a sense that can be made very precise via the considerations developed in Section 2 below. It was then natural to ask for a similar coding of the genealogy of more general continuous-state branching processes. Recall that continuous-state branching processes are Markov processes with values in \mathbb{R}_+ , which are the possible limits of rescaled Galton-Watson branching processes (Lamperti [40]). Such processes are characterized by a branching mechanism function ψ , with $\psi(u) = cu^2$ in the case of Feller's diffusion. The problem of coding the genealogy of general (critical or subcritical) continuous-state branching processes was treated in two papers of Le Gall and Le Jan [47], [48] (see also the monograph [21]). The role of the Brownian excursion in the case of Feller's diffusion is played by the so-called height process, which is a (non-Markovian) function of the Lévy process with no negative jumps and Laplace exponent ψ . The construction of the height process and its relations with the genealogy of continuous-state branching processes made it possible to investigate the asymptotics of critical Galton-Watson trees when the offspring distribution has infinite variance (see Chapter 2 of [21], and [20]). See Section 4 below for a brief presentation in the stable case where $\psi(u) = u^\alpha$ for some $1 < \alpha \leq 2$.

In the present work, we give a survey of the preceding results, and of some recent applications, using the language and the formalism of real trees (cf Section 1). Although real trees have been studied for a long time for algebraic or geometric purposes (see e.g. [18] and [55]), their use in probability theory seems to be quite recent. The CRT is naturally viewed as a random real tree (see Section 3 below), but this interpretation was not made explicit in Aldous' work. The recent paper [27] of Evans, Pitman and Winter starts with a general study of real trees from the point of view of measure theory, and establishes in particular that the space \mathbb{T} of equivalent classes of (rooted) compact real trees, endowed with the Gromov-Hausdorff metric, is a Polish space. This makes it very natural to consider random variables or even random processes taking values in the space \mathbb{T} ([27] gives a particularly nice example of such a process, which combines root growth and regrafting and converges in distribution to the CRT, see also [28] for further developments along these lines).

Our presentation owes a lot to the recent paper [22], which uses the formalism of real trees to define the so-called Lévy trees that were implicit in [47] or [21], and to study various probabilistic and fractal properties of these objects. In particular, the coding of a real tree by a (deterministic) excursion is made precise in Section 2 below, which is taken from Section 2 of [22]. Aldous' theorem relating the CRT to discrete Galton-Watson trees, and other analogous results involving the more general stable trees can be restated elegantly in the framework of real trees (see Sections 3 and 4 below). Stable trees have a number of remarkable properties, some of which are briefly presented in Sections 4 and 5. In particular, they possess a nice self-similarity property (Proposition 4.3) and they also verify a branching property (Theorem 5.1) that is analogous to the well-known branching property of Galton-Watson trees: Conditionally given the tree below level $a > 0$, the subtrees originating from that level are distributed as the atoms of a Poisson point measure whose intensity involves a local time measure supported on the vertices at distance a from the root. These local

times make it possible to define the uniform measure on the tree in an intrinsic way. Section 6 shows that the Lévy snake construction of superprocesses, which generalizes the Brownian snake approach, takes a neat form in the formalism of real trees (Theorem 6.1). Finally, in Section 7, we give a very recent application of these concepts to asymptotics for random quadrangulations, which involves a particular conditioning of our random real trees.

1 Real trees

We start with a basic definition (see e.g. [18]).

Definition 1.1 *A metric space (\mathcal{T}, d) is a real tree if the following two properties hold for every $\sigma_1, \sigma_2 \in \mathcal{T}$.*

- (i) *There is a unique isometric map f_{σ_1, σ_2} from $[0, d(\sigma_1, \sigma_2)]$ into \mathcal{T} such that $f_{\sigma_1, \sigma_2}(0) = \sigma_1$ and $f_{\sigma_1, \sigma_2}(d(\sigma_1, \sigma_2)) = \sigma_2$.*
- (ii) *If q is a continuous injective map from $[0, 1]$ into \mathcal{T} , such that $q(0) = \sigma_1$ and $q(1) = \sigma_2$, we have*

$$q([0, 1]) = f_{\sigma_1, \sigma_2}([0, d(\sigma_1, \sigma_2)]).$$

A rooted real tree is a real tree (\mathcal{T}, d) with a distinguished vertex $\rho = \rho(\mathcal{T})$ called the root.

Remark We use the terminology real tree rather than \mathbb{R} -tree as in [18] or [55].

In what follows, real trees will always be rooted and compact, even if this is not mentioned explicitly.

Let us consider a rooted real tree (\mathcal{T}, d) . The range of the mapping f_{σ_1, σ_2} in (i) is denoted by $[[\sigma_1, \sigma_2]]$ (this is the line segment between σ_1 and σ_2 in the tree). In particular, for every $\sigma \in \mathcal{T}$, $[[\rho, \sigma]]$ is the path going from the root to σ , which we will interpret as the ancestral line of vertex σ . More precisely we can define a partial order on the tree by setting $\sigma \preceq \sigma'$ (σ is an ancestor of σ') if and only if $\sigma \in [[\rho, \sigma']]$. If $\sigma, \sigma' \in \mathcal{T}$, there is a unique $\eta \in \mathcal{T}$ such that $[[\rho, \sigma]] \cap [[\rho, \sigma']] = [[\rho, \eta]]$. We write $\eta = \sigma \wedge \sigma'$ and call η the most recent common ancestor to σ and σ' .

By definition, the multiplicity $\mathbf{k}(\sigma)$ of a vertex $\sigma \in \mathcal{T}$ is the number of connected components of $\mathcal{T} \setminus \{\sigma\}$. Vertices of $\mathcal{T} \setminus \{\rho\}$ which have multiplicity 1 are called leaves.

Let us now discuss convergence of real trees. Two rooted real trees $\mathcal{T}_{(1)}$ and $\mathcal{T}_{(2)}$ are called equivalent if there is a root-preserving isometry that maps $\mathcal{T}_{(1)}$ onto $\mathcal{T}_{(2)}$. We denote by \mathbb{T} the set of all equivalence classes of rooted compact real trees. The set \mathbb{T} can be equipped with the (pointed) Gromov-Hausdorff distance, which is defined as follows.

If (E, δ) is a metric space, we use the notation $\delta_{Haus}(K, K')$ for the usual Hausdorff metric between compact subsets of E . Then, if \mathcal{T} and \mathcal{T}' are two

rooted compact real trees with respective roots ρ and ρ' , we define the distance $d_{GH}(\mathcal{T}, \mathcal{T}')$ as

$$d_{GH}(\mathcal{T}, \mathcal{T}') = \inf \left(\delta_{Haus}(\varphi(\mathcal{T}), \varphi'(\mathcal{T}')) \vee \delta(\varphi(\rho), \varphi'(\rho')) \right),$$

where the infimum is over all isometric embeddings $\varphi : \mathcal{T} \rightarrow E$ and $\varphi' : \mathcal{T}' \rightarrow E$ of \mathcal{T} and \mathcal{T}' into a common metric space (E, δ) . Obviously $d_{GH}(\mathcal{T}, \mathcal{T}')$ only depends on the equivalence classes of \mathcal{T} and \mathcal{T}' . Furthermore d_{GH} defines a metric on \mathbb{T} (cf [32] and [27]).

Theorem 1.1 ([27]) *The metric space (\mathbb{T}, d_{GH}) is complete and separable.*

Furthermore, the distance d_{GH} can often be evaluated in the following way. First recall that if (E_1, d_1) and (E_2, d_2) are two compact metric spaces, a correspondence between E_1 and E_2 is a subset \mathcal{R} of $E_1 \times E_2$ such that for every $x_1 \in E_1$ there exists at least one $x_2 \in E_2$ such that $(x_1, x_2) \in \mathcal{R}$ and conversely for every $y_2 \in E_2$ there exists at least one $y_1 \in E_1$ such that $(y_1, y_2) \in \mathcal{R}$. The distortion of the correspondence \mathcal{R} is defined by

$$\text{dis}(\mathcal{R}) = \sup\{|d_1(x_1, y_1) - d_2(x_2, y_2)| : (x_1, x_2), (y_1, y_2) \in \mathcal{R}\}.$$

Then, if \mathcal{T} and \mathcal{T}' are two rooted \mathbb{R} -trees with respective roots ρ and ρ' , we have

$$d_{GH}(\mathcal{T}, \mathcal{T}') = \frac{1}{2} \inf_{\mathcal{R} \in \mathcal{C}(\mathcal{T}, \mathcal{T}'), (\rho, \rho') \in \mathcal{R}} \text{dis}(\mathcal{R}), \quad (1)$$

where $\mathcal{C}(\mathcal{T}, \mathcal{T}')$ denotes the set of all correspondences between \mathcal{T} and \mathcal{T}' (see Lemma 2.3 in [27]).

2 Coding compact real trees

Our main goal in this section is to describe a method for constructing real trees, which is particularly well-suited to our forthcoming applications to random trees. We consider a (deterministic) continuous function $g : [0, \infty) \rightarrow [0, \infty)$ with compact support and such that $g(0) = 0$. To avoid trivialities, we will also assume that g is not identically zero. For every $s, t \geq 0$, we set

$$m_g(s, t) = \inf_{r \in [s \wedge t, s \vee t]} g(r),$$

and

$$d_g(s, t) = g(s) + g(t) - 2m_g(s, t).$$

Clearly $d_g(s, t) = d_g(t, s)$ and it is also easy to verify the triangle inequality

$$d_g(s, u) \leq d_g(s, t) + d_g(t, u)$$

for every $s, t, u \geq 0$. We then introduce the equivalence relation $s \sim t$ iff $d_g(s, t) = 0$ (or equivalently iff $g(s) = g(t) = m_g(s, t)$). Let \mathcal{T}_g be the quotient space

$$\mathcal{T}_g = [0, \infty) / \sim.$$

Obviously the function d_g induces a distance on \mathcal{T}_g , and we keep the notation d_g for this distance. We denote by $p_g : [0, \infty) \rightarrow \mathcal{T}_g$ the canonical projection. Clearly p_g is continuous (when $[0, \infty)$ is equipped with the Euclidean metric and \mathcal{T}_g with the metric d_g).

Theorem 2.1 ([22]) *The metric space (\mathcal{T}_g, d_g) is a real tree.*

We will always view (\mathcal{T}_g, d_g) as a rooted real tree with root $\rho = p_g(0)$. If $\zeta > 0$ is the supremum of the support of g , we have $p_g(t) = \rho$ for every $t \geq \zeta$. In particular, $\mathcal{T}_g = p_g([0, \zeta])$ is compact. We will call \mathcal{T}_g the real tree coded by g .

A useful ingredient in the proof of Theorem 2.1 is the following root-change lemma, whose proof is an elementary exercise.

Lemma 2.2 *Let $s_0 \in [0, \zeta)$. For any real $r \geq 0$, denote by \overline{r} the unique element of $[0, \zeta)$ such that $r - \overline{r}$ is an integer multiple of ζ . Set*

$$g'(s) = g(s_0) + g(\overline{s_0 + s}) - 2m_g(s_0, \overline{s_0 + s}),$$

for every $s \in [0, \zeta]$, and $g'(s) = 0$ for $s > \zeta$. Then, the function g' is continuous with compact support and satisfies $g'(0) = 0$, so that we can define the metric space $(\mathcal{T}_{g'}, d_{g'})$. Furthermore, for every $s, t \in [0, \zeta]$, we have

$$d_{g'}(s, t) = d_g(\overline{s_0 + s}, \overline{s_0 + t}) \quad (2)$$

and there exists a unique isometry R from $\mathcal{T}_{g'}$ onto \mathcal{T}_g such that, for every $s \in [0, \zeta]$,

$$R(p_{g'}(s)) = p_g(\overline{s_0 + s}). \quad (3)$$

Thanks to the lemma, the fact that \mathcal{T}_g verifies property (i) in the definition of a real tree is obtained from the particular case when $\sigma_1 = \rho$ and $\sigma_2 = \sigma = p_g(s)$ for some $s \in [0, \zeta]$. In that case however, the isometric mapping $f_{\rho, \sigma}$ is easily constructed by setting

$$f_{\rho, \sigma}(t) = p_g(\sup\{r \leq s : g(r) = t\}) , \quad \text{for every } 0 \leq t \leq g(s) = d_g(\rho, \sigma).$$

The remaining part of the argument is straightforward: See Section 2 in [22] (or Lemma 3.1 in [28] for a different argument).

Remark In addition to the natural genealogical order, the tree \mathcal{T}_g is also equipped with the total order induced by the order on the real line and the coding function g . We will not use this observation here as we prefer to consider real trees as unordered trees (compare with the discrete plane trees that are discussed in Section 3).

Can one compare the trees coded by two different functions g and g' ? The following lemma gives a simple estimate.

Lemma 2.3 *Let g and g' be two continuous functions with compact support from $[0, \infty)$ into $[0, \infty)$, such that $g(0) = g'(0) = 0$. Then,*

$$d_{GH}(\mathcal{T}_g, \mathcal{T}_{g'}) \leq 2\|g - g'\|,$$

where $\|g - g'\|$ stands for the supremum norm of $g - g'$.

Proof. We can construct a correspondence between \mathcal{T}_g and $\mathcal{T}_{g'}$ by setting

$$\mathcal{R} = \{(\sigma, \sigma') : \sigma = p_g(t) \text{ and } \sigma' = p_{g'}(t) \text{ for some } t \geq 0\}.$$

In order to bound the distortion of \mathcal{R} , let $(\sigma, \sigma') \in \mathcal{R}$ and $(\eta, \eta') \in \mathcal{R}$. By our definition of \mathcal{R} we can find $s, t \geq 0$ such that $p_g(s) = \sigma$, $p_{g'}(s) = \sigma'$ and $p_g(t) = \eta$, $p_{g'}(t) = \eta'$. Now recall that $d_g(\sigma, \eta) = g(s) + g(t) - 2m_g(s, t)$ and $d_{g'}(\sigma', \eta') = g'(s) + g'(t) - 2m_{g'}(s, t)$, so that

$$|d_g(\sigma, \eta) - d_{g'}(\sigma', \eta')| \leq 4\|g - g'\|.$$

Hence we have $\text{dis}(\mathcal{R}) \leq 4\|g - g'\|$ and the desired result follows from (1). ■

The mapping $g \rightarrow \mathcal{T}_g$ is thus continuous if the set of functions satisfying our assumptions is equipped with the supremum norm. In particular, this mapping is measurable.

3 Galton-Watson trees and the CRT

Denote by $(\mathbf{e}_t)_{0 \leq t \leq 1}$ a normalized Brownian excursion. Informally, $(\mathbf{e}_t)_{0 \leq t \leq 1}$ is just a linear Brownian path started from the origin and conditioned to stay positive over the time interval $(0, 1)$, and to come back to 0 at time 1. See e.g. Sections 2.9 and 2.10 of Itô and McKean [36] for a discussion of the normalized excursion. We extend the definition of \mathbf{e}_t by setting $\mathbf{e}_t = 0$ if $t > 1$. Then the (random) function \mathbf{e} satisfied the assumptions of the previous section and we can thus consider the real tree $\mathcal{T}_{\mathbf{e}}$, which is a random variable with values in \mathbb{T} .

Definition 3.1 *The random real tree $\mathcal{T}_{\mathbf{e}}$ is called the Continuum Random Tree (CRT).*

The CRT was initially defined by Aldous [1] with a different formalism, but the preceding definition corresponds to Corollary 22 in [3], up to an unimportant scaling factor 2.

One major motivation for studying the CRT is the fact that it occurs as the scaling limit of critical Galton-Watson trees conditioned to have a large (fixed) number of vertices. In order to state this result properly, we need to introduce some formalism for discrete trees, which we borrow from Neveu [54]. Let

$$\mathcal{U} = \bigcup_{n=0}^{\infty} \mathbb{N}^n$$

where $\mathbb{N} = \{1, 2, \dots\}$ and by convention $\mathbb{N}^0 = \{\emptyset\}$. If $u = (u_1, \dots, u_m)$ and $v = (v_1, \dots, v_n)$ belong to \mathcal{U} , we write $uv = (u_1, \dots, u_m, v_1, \dots, v_n)$ for the concatenation of u and v . In particular $u\emptyset = \emptyset u = u$.

Definition 3.2 A plane tree θ is a finite subset of \mathcal{U} such that:

- (i) $\emptyset \in \theta$.
- (ii) If $v \in \theta$ and $v = uj$ for some $u \in \mathcal{U}$ and $j \in \mathbb{N}$, then $u \in \theta$.
- (iii) For every $u \in \theta$, there exists a number $k_u(\theta) \geq 0$ such that $uj \in \theta$ if and only if $1 \leq j \leq k_u(\theta)$.

We denote by \mathbf{T} the set of all plane trees. In what follows, we see each vertex of the tree as an individual of a population whose θ is the family tree.

If θ is a tree and $u \in \theta$, we define the shift of θ at u by $\tau_u\theta = \{v \in \mathcal{U} : uv \in \theta\}$. Note that $\tau_u\theta \in \mathbf{T}$. We also denote by $h(\theta)$ the height of θ , that is the maximal generation of a vertex in θ , and by $\#\theta$ the number of vertices of θ .

For our purposes it will be convenient to view θ as a real tree: To this end, embed θ in the plane, in such a way that each edge corresponds to a line segment of length one, in the way suggested by the left part of Fig. 1. Denote by \mathcal{T}^θ the union of all these line segments and equip \mathcal{T}^θ with the obvious metric such that the distance between σ and σ' is the length of the shortest path from σ to σ' in \mathcal{T}^θ . This construction leads to a (compact rooted) real tree whose equivalence class does not depend on the particular embedding.

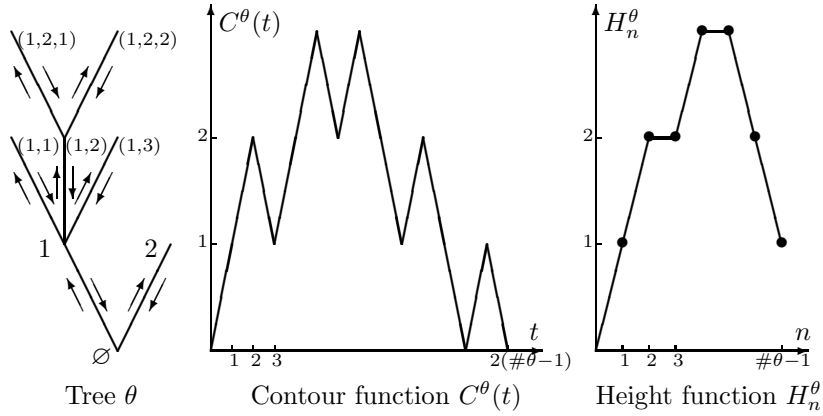


Figure 1

The discrete tree θ , or equivalently the tree \mathcal{T}^θ , can be coded by two simple discrete functions, namely the *contour function* and the *height function*. To define the contour function, consider a particle that starts from the root of \mathcal{T}^θ and visits continuously all edges at speed one, going backwards as less as possible and respecting the lexicographical order of vertices. Then let $C^\theta(t)$ denote the distance to the root of the position of the particle at time t (for $t \geq 2(\#\theta - 1)$, we take $C^\theta(t) = 0$ by convention). Fig.1 explains the definition of the contour function better than a formal definition. Note that in the notation of Section

2, we have $\mathcal{T}^\theta = \mathcal{T}_{C^\theta}$, meaning that \mathcal{T}^θ coincides with the tree coded by the function C^θ .

The height function $H^\theta = (H_n^\theta, 0 \leq n < \#\theta)$ is a discrete function defined as follows. Write $u_0 = \emptyset, u_1 = 1, u_2, \dots, u_{\#\theta-1}$ for the elements of θ listed in lexicographical order. Then H_n^θ is the generation of u_n (cf Fig.1).

Now let us turn to Galton-Watson trees. Let μ be a critical offspring distribution. This means that μ is a probability measure on \mathbb{Z}_+ such that $\sum_{k=0}^{\infty} k\mu(k) = 1$. We exclude the trivial case where $\mu(1) = 1$. Then, there is a unique probability distribution Π_μ on \mathbf{T} such that

- (i) $\Pi_\mu(k_\emptyset = j) = \mu(j), \quad j \in \mathbb{Z}_+.$
- (ii) For every $j \geq 1$ with $\mu(j) > 0$, the shifted trees $\tau_1\theta, \dots, \tau_j\theta$ are independent under the conditional probability $\Pi_\mu(\cdot \mid k_\emptyset = j)$ and their conditional distribution is Π_μ .

A random tree with distribution Π_μ is called a Galton-Watson tree with offspring distribution μ , or in short a μ -Galton-Watson tree. Obviously it describes the genealogy of the Galton-Watson process with offspring distribution μ started initially with one individual.

We are now able to state our invariance principle for Galton-Watson trees. If \mathcal{T} is a real tree with metric d and $r > 0$, we write $r\mathcal{T}$ for the “same” tree with metric rd . We say that μ is aperiodic if it is not supported on a proper subgroup of \mathbb{Z} .

Theorem 3.1 *Assume that the offspring distribution μ is critical with finite variance $\sigma^2 > 0$, and is aperiodic. Then the distribution of the rescaled tree $\frac{\sigma}{2\sqrt{n}}\mathcal{T}^\theta$ under the probability measure $\Pi_\mu(\cdot \mid \#\theta = n)$ converges as $n \rightarrow \infty$ to the law of the CRT.*

This is an immediate consequence of Theorem 23 in [3]. In fact the latter result states that the rescaled contour function $(\frac{\sigma}{2\sqrt{n}}C^\theta(2nt), 0 \leq t \leq 1)$ under $\Pi_\mu(\cdot \mid \#\theta = n)$ converges in distribution to the normalized Brownian excursion e . Note that the rescaled tree $\frac{\sigma}{2\sqrt{n}}\mathcal{T}^\theta$ is coded by the function $\frac{\sigma}{2\sqrt{n}}C^\theta$, in the sense of Section 2 above. It then suffices to apply Lemma 2.3.

A simple approach to the convergence of the rescaled contour functions towards the normalized Brownian excursion was provided in [51]. A key ingredient is the fact that the height function under Π_μ can be written as a simple functional of a random walk. Let S be a random walk with jump distribution $\nu(i) = \mu(i+1)$ ($i = -1, 0, 1, 2, \dots$) started from the origin, and $T = \inf\{n \geq 1 : S_n = -1\}$. Then, under Π_μ ,

$$(H_n^\theta, 0 \leq n < \#\theta) \stackrel{(d)}{=} (K_n, 0 \leq n < T)$$

where

$$K_n = \text{Card}\left\{0 \leq j < n : S_j = \inf_{j \leq k \leq n} S_k\right\}. \quad (4)$$

Formula (4) was noticed in [47], where it motivated the coding of generalizations of the CRT (see Section 4 below). Under the additional assumption that μ has exponential moments, formula (4) was used in [51] to show that the height function (and then the contour function) under $\Pi_\mu(\cdot \mid \#\theta = n)$ is close to an excursion of the random walk S of length n . Invariance principles relating random walk excursions to Brownian excursions then lead to the desired result.

See also Bennies and Kersting [7] for a nice elementary presentation of the relations between Galton-Watson trees and random walks, with an application to a (weak) version of Theorem 3.1, and Chapters 5 and 6 of Pitman [58].

Remarks (a) Theorem 3.1 has various applications to the asymptotic behavior of functionals of Galton-Watson trees. For instance, by considering the height of trees, we easily get that for every $x \geq 0$,

$$\lim_{n \rightarrow \infty} \Pi_\mu \left(h(\theta) \geq \frac{2x\sqrt{n}}{\sigma} \mid \#\theta = n \right) = P(M(\mathbf{e}) \geq x), \quad (5)$$

where

$$M(\mathbf{e}) = \sup_{s \geq 0} \mathbf{e}(s).$$

The probability in the right-hand side of (5) is known in the form of a series (Kennedy [37], see also Chung [14] for related results, and Section 3.1 in [2]).

(b) Special choices of the offspring distribution μ lead to limit theorems for “combinatorial trees”. For instance, if we let μ be the geometric distribution $\mu(k) = 2^{-k-1}$, which satisfies all our assumptions with $\sigma^2 = 2$, then $\Pi_\mu(\cdot \mid \#\theta = n)$ is easily identified as the uniform distribution on the set of all plane trees with n vertices. In particular, (5) gives the asymptotic proportion of those plane trees with n vertices which have a height greater than $x\sqrt{2n}$. Similar observations apply to other classes of discrete trees, e.g. to binary trees (take $\mu(0) = \mu(2) = 1/2$, and note that we need a slight extension of Theorem 3.1, since μ is not aperiodic), or to Cayley trees (corresponding to the case when μ is Poisson with parameter 1). Such asymptotics had in fact been established before Theorem 3.1 was proved, by the method of generating functions: See [12] and especially Flajolet and Odlyzko [29].

(c) The convergence in Theorem 3.1 is not strong enough to allow one to deal with all interesting functionals of the tree. Still, Theorem 3.1 can be used to guess the kind of limit theorem one should expect. A typical example is the height profile of the tree, that is for every level k the number of vertices at generation k . Theorem 3.1 strongly suggests that the suitably rescaled height profile of a Galton-Watson tree conditioned to have n vertices should converge in distribution towards the local time process of a normalized Brownian excursion (see also the discussion of local times of stable trees in Section 5 below). This was indeed proved by Drmota and Gittenberger [19] (see also Pitman [57] and Aldous [5]).

The effect of conditioning on the event $\{\#\theta = n\}$ is to force the tree θ to be large. One can imagine various other conditionings that have the same effect,

and will give rise to different limiting real trees. Typically, these limits will be described in terms of the Itô excursion measure. Recall that the Itô excursion measure $n(de)$ is the σ -finite measure on the space $C(\mathbb{R}_+, \mathbb{R}_+)$ of continuous functions from \mathbb{R}_+ and \mathbb{R}_+ , which can be obtained as

$$n(de) = \lim_{\varepsilon \rightarrow 0} \frac{1}{2\varepsilon} P_\varepsilon(de) \quad (6)$$

where $P_\varepsilon(de)$ stand for the distribution of a linear Brownian motion started from ε and stopped at the first time when it hits 0, and we omit the precise meaning of the convergence (6) (see Chapter XII of [59] for a thorough discussion of Itô's excursion measure). We write $\zeta(e) = \inf\{s > 0 : e(s) = 0\}$ for the duration of e . The connection with the normalized Brownian excursion is made by noting that the distribution of \mathbf{e} is just the conditioned measure $n(de \mid \zeta(e) = 1)$.

We write $\Theta_2(dT)$ for the distribution of the tree \mathcal{T}_e under $n(de)$ (the reason for the subscript 2 will become clear later). Self-similarity properties of our random real trees become apparent on the measure Θ_2 .

Proposition 3.2 *For every $r > 0$, the distribution of $r\mathcal{T}$ under Θ_2 is $r\Theta_2$.*

This result readily follows from the analogous statement for the Itô measure.

To illustrate the usefulness of introducing the measure Θ , we state a variant of Theorem 3.1. This variant and other results of the same type can be found in Chapter 2 of [21] (see also Geiger and Kersting [31] for related results). The height $h(\mathcal{T})$ of a real tree is obviously defined as the maximal distance to the root.

Theorem 3.3 *Under the same assumptions as in Theorem 3.1, the distribution of the tree $n^{-1/2}\mathcal{T}^\theta$ under $\Pi_\mu(d\theta \mid h(\theta) > n^{1/2})$ converges as $n \rightarrow \infty$ to $\Theta_2(dT \mid h(\mathcal{T}) > 1)$.*

In contrast with Theorem 3.1, this statement does not involve the constant σ .

4 Stable trees

One may ask what happens in Theorems 3.1 and 3.3 in the case when μ has infinite variance. Assuming that μ is in the domain of attraction of a stable distribution with index $\alpha \in (1, 2)$, we still get a limiting random real tree, which is called the stable tree with index α .

Before stating this result, we need to spend some time defining the limiting object. We fix $\alpha \in (1, 2)$ and consider a stable Lévy process $X = (X_t)_{t \geq 0}$ with index α and no negative jumps, started from the origin. The absence of negative jumps implies that $E[\exp(-\lambda X_t)] < \infty$ for every $\lambda \geq 0$, and we may normalize X so that

$$E[\exp(-\lambda X_t)] = \exp(\lambda^\alpha t).$$

Set $I_t = \inf_{r \leq t} X_r$, which is a continuous process. Then it is well known that the process $X - I$ is Markovian. Furthermore, the point 0 is regular for $X - I$, and the process $-I$ provides a local time at the origin for this Markov process. The associated excursion measure is denoted by \mathbf{N} . It will play the same role as the Itô excursion measure (which one recovers in the case $\alpha = 2$, up to inimportant scaling constants) in the previous paragraph. We again use the notation ζ for the duration of the excursion under \mathbf{N} .

Proposition 4.1 *There exists a continuous process $(H_s)_{s \geq 0}$, called the height process, such that*

$$H_s = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \int_0^s dr \mathbf{1} \left(X_r \leq \inf_{r \leq u \leq s} X_u + \varepsilon \right) \quad , \quad \text{for every } s \geq 0, \quad \mathbf{N} \text{ a.e.}$$

Moreover $H_0 = 0$ and $H_s = 0$ for every $s \geq \zeta$, \mathbf{N} a.e.

The formula for H_s is a continuous analogue of the formula (4) for the discrete height function of a Galton-Watson tree.

Definition 4.1 *The measure Θ_α defined as the law of the tree \mathcal{T}_H under \mathbf{N} , is called the distribution of the stable tree with index α . We also denote by $\Theta_\alpha^{(1)}$ the distribution of \mathcal{T}_H under the probability measure $\mathbf{N}(\cdot \mid \zeta = 1)$.*

We can now state an analogue of Theorem 3.1 in the stable case.

Theorem 4.2 ([20]) *Suppose that μ is critical and aperiodic, and that it is in the domain of attraction of the stable law with index $\alpha \in (1, 2)$, meaning that there exists a sequence $a_n \uparrow \infty$ such that, if Y_1, Y_2, \dots are i.i.d. with distribution μ ,*

$$\frac{1}{a_n} (Y_1 + \dots + Y_n - n) \xrightarrow[n \rightarrow \infty]{(d)} X_1.$$

Then, the law of the tree $n^{-1}a_n \mathcal{T}^\theta$ under $\Pi_\mu(\cdot \mid \#\theta = n)$ converges as $n \rightarrow \infty$ to $\Theta_\alpha^{(1)}$.

Other limit theorems relating discrete Galton-Watson trees to the measures Θ_α , in the spirit of Theorem 3.3, can be found in Chapter 2 of [21] and in [22] (Theorem 4.1). These results apply more generally to the Lévy trees of [22], which can be viewed as possible limits of sequences of critical Galton-Watson trees for which the offspring distribution depends on the tree taken in the sequence (in a way very similar to the classical approximations of Lévy processes by random walks). The recent article [30] gives a limit theorem for Galton-Watson trees with possibly infinite variance, which is related to the results of [20] and [21].

The self-similarity property of Proposition 3.2 extends to stable trees:

Proposition 4.3 *For every $r > 0$, the distribution of $r\mathcal{T}$ under Θ_α is $r^{\frac{1}{\alpha-1}} \Theta_\alpha$.*

5 Probabilistic and fractal properties of stable trees

In this section we give some important properties of our stable trees. The following results, which are taken from [22], hold for any $\alpha \in (1, 2]$.

Consider first a fixed real tree (\mathcal{T}, d) in \mathbb{T} with root $\rho(\mathcal{T})$. The level set of \mathcal{T} at level $a > 0$ is

$$\mathcal{T}(a) = \{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) = a\}.$$

The truncation of the tree \mathcal{T} at level a is the new tree

$$\text{tr}_a(\mathcal{T}) = \{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) \leq a\},$$

which is obviously equipped with the restriction of the distance d . It is easy to verify that the mapping $\mathcal{T} \rightarrow \text{tr}_a(\mathcal{T})$ from \mathbb{T} into itself is measurable.

Let us fix $a > 0$ and denote by $\mathcal{T}^{(i), \circ}$, $i \in \mathcal{I}$ the connected components of the open set

$$\mathcal{T}((a, \infty)) = \{\sigma \in \mathcal{T} : d(\rho(\mathcal{T}), \sigma) > a\}.$$

Notice that the index set \mathcal{I} may be empty (if $h(\mathcal{T}) \leq a$), finite or countable. Let $i \in \mathcal{I}$. Then the ancestor of σ at level a must be the same for every $\sigma \in \mathcal{T}^{(i), \circ}$. We denote by σ_i this common ancestor and set $\mathcal{T}^{(i)} = \mathcal{T}^{(i), \circ} \cup \{\sigma_i\}$. Then $\mathcal{T}^{(i)}$ is a compact rooted \mathbb{R} -tree with root σ_i . The trees $\mathcal{T}^{(i)}$, $i \in \mathcal{I}$ are called the subtrees of \mathcal{T} originating from level a , and we set

$$\mathcal{N}_a^{\mathcal{T}} := \sum_{i \in \mathcal{I}} \delta_{(\sigma_i, \mathcal{T}^{(i)})},$$

which is a point measure on $\mathcal{T}(a) \times \mathbb{T}$.

Theorem 5.1 *For every $a > 0$ and for Θ_α a.e. $\mathcal{T} \in \mathbb{T}$ we can define a finite measure ℓ^a on \mathcal{T} , in such a way that the following properties hold:*

- (i) *For every $a > 0$, ℓ^a is supported on $\mathcal{T}(a)$, and $\{\ell^a \neq 0\} = \{h(\mathcal{T}) > a\}$, $\Theta_\alpha(d\mathcal{T})$ a.e.*
- (ii) *For every $a > 0$, we have $\Theta_\alpha(d\mathcal{T})$ a.e. for every bounded continuous function φ on \mathcal{T} ,*

$$\begin{aligned} \langle \ell^a, \varphi \rangle &= \lim_{\varepsilon \downarrow 0} C_\alpha \varepsilon^{\frac{1}{\alpha-1}} \int \mathcal{N}_a^{\mathcal{T}}(d\sigma d\mathcal{T}') \varphi(\sigma) \mathbf{1}_{\{h(\mathcal{T}') \geq \varepsilon\}} \\ &= \lim_{\varepsilon \downarrow 0} C_\alpha \varepsilon^{\frac{1}{\alpha-1}} \int \mathcal{N}_{a-\varepsilon}^{\mathcal{T}}(d\sigma d\mathcal{T}') \varphi(\sigma) \mathbf{1}_{\{h(\mathcal{T}') \geq \varepsilon\}} \end{aligned} \quad (7)$$

where $C_\alpha = (\alpha - 1)^{1/(\alpha-1)}$ if $1 < \alpha < 2$ and $C_2 = 2$. Furthermore, for every $a > 0$, the conditional distribution of the point measure $\mathcal{N}_a^{\mathcal{T}}(d\sigma d\mathcal{T}')$, under the probability measure $\Theta_\alpha(d\mathcal{T} \mid h(\mathcal{T}) > a)$ and given $\text{tr}_a(\mathcal{T})$, is that of a Poisson point measure on $\mathcal{T}(a) \times \mathbb{T}$ with intensity $\ell^a(d\sigma)\Theta_\alpha(d\mathcal{T}')$.

The last property is the most important one. It may be called the *branching property* of the stable tree as it is exactly analogous to the classical branching property for Galton-Watson trees (cf Property (ii) in the definition of Galton-Watson trees in Section 3). The random measure ℓ^a will be called the *local time* of \mathcal{T} at level a . Note that the normalization of local times, that is the particular choice of the constant C_α in (7), is made so that the branching property holds in the form given in the theorem.

Remarks (a) The branching property holds for the more general Lévy trees which are considered in [22]. Other classes of random real trees, which are related to various problems of probability theory or combinatorics, are studied in [6] and [33]. A form of the branching property was used by Miermont [53] in his study of fragmentations of the stable tree.

(b) It is possible to choose a modification of the collection $(\ell^a, a > 0)$ which has good continuity properties. Precisely, this process has a càdlàg modification, which is even continuous when $\alpha = 2$. When $\alpha < 2$, the discontinuity points of the mapping $a \rightarrow \ell^a$ correspond to levels of points of infinite multiplicity of the tree (cf Proposition 5.2 below).

Local times can be used to give an intrinsic definition of the uniform measure on the tree: We set

$$\mathbf{m} = \int_0^\infty da \ell^a.$$

Note that if the tree is constructed as $\mathcal{T} = \mathcal{T}_H$ (resp. $\mathcal{T} = \mathcal{T}_e$ when $\alpha = 2$), the measure \mathbf{m} corresponds to the image measure of Lebesgue measure on $[0, \zeta]$ under the coding function $s \rightarrow p_H(s)$ (resp. $s \rightarrow p_e(s)$). Thanks to this observation, the probability measure $\Theta_\alpha^{(1)}$ can be identified with $\Theta_\alpha(\cdot \mid \mathbf{m}(\mathcal{T}) = 1)$.

The next proposition gives precise information about the multiplicity of vertices in our stable trees. Recall that $\mathbf{k}(\sigma)$ denotes the multiplicity of σ .

Proposition 5.2 *We have $\mathbf{m}(\{\sigma \in \mathcal{T} : \mathbf{k}(\sigma) > 1\}) = 0$, Θ_α a.e. (in other words almost all vertices are leaves). Moreover, we have Θ_α a.e.:*

- (i) *If $\alpha = 2$, $\mathbf{k}(\sigma) \in \{1, 2, 3\}$ for all $\sigma \in \mathcal{T}$, and the set $\{\sigma \in \mathcal{T} : \mathbf{k}(\sigma) = 3\}$ of binary branching points is a countable dense subset of \mathcal{T} .*
- (ii) *If $1 < \alpha < 2$, $\mathbf{k}(\sigma) \in \{1, 2, \infty\}$ for all $\sigma \in \mathcal{T}$, and the set $\{\sigma \in \mathcal{T} : \mathbf{k}(\sigma) = \infty\}$ of infinite branching points is a countable dense subset of \mathcal{T} .*

Stable trees also enjoy a nice invariance property under uniform re-rooting, which is related to the deterministic re-rooting Lemma 2.2. We state this property under the law $\Theta_\alpha^{(1)}$ of the normalized tree. If \mathcal{T} is a tree and $\sigma \in \mathcal{T}$, we write $\mathcal{T}^{[\sigma]}$ for the “same” tree \mathcal{T} with root σ .

Proposition 5.3 *The law of the tree $\mathcal{T}^{[\sigma]}$ under the measure $\Theta_\alpha^{(1)}(d\mathcal{T}) \mathbf{m}(d\sigma)$ coincides with $\Theta_\alpha^{(1)}(d\mathcal{T})$.*

In the case $\alpha = 2$, this invariance property was already noticed in Aldous [2]. Still for $\alpha = 2$, more precise invariance properties under re-rooting can be found in [52] and [50].

We conclude this section with a discussion of the fractal dimension of stable trees. If E is a subset of \mathbb{R}_+ , we use the notation

$$\mathcal{T}(E) = \{\sigma \in \mathcal{T} : d(\rho, \sigma) \in E\}$$

where $\rho = \rho(\mathcal{T})$ is the root of \mathcal{T} .

Theorem 5.4 *Let E be a compact subset of $(0, \infty)$ and $A = \sup E$. Assume that the Hausdorff and upper box dimensions of E are equal and let $d(E) \in [0, 1]$ be their common value. Then, Θ_α a.e. on the event $\{h(\mathcal{T}) > A\}$, the Hausdorff and packing dimensions of $\mathcal{T}(E)$ coincide and are equal to*

$$\dim(\mathcal{T}(E)) = d(E) + \frac{1}{\alpha - 1}.$$

In particular,

$$\dim(\mathcal{T}) = \frac{\alpha}{\alpha - 1} \quad , \quad \Theta_\alpha \text{ a.e.}$$

and, for every $a > 0$,

$$\dim(\mathcal{T}(a)) = \frac{1}{\alpha - 1} \quad , \quad \Theta_\alpha \text{ a.e. on } \{h(\mathcal{T}) > a\}.$$

Remark The formula for $\dim(\mathcal{T})$ has also been derived by Haas and Miermont [33] independently of [22]. More precise information about the Hausdorff measure of stable trees can be found in [23].

6 Spatial trees

We will now explain how the genealogical structure of our stable trees can be combined with spatial displacements (given by independent Brownian motions in \mathbb{R}^k) to yield a construction of superprocesses with a stable branching mechanism. To present this construction in a way suitable for applications, it is convenient to introduce the notion of a spatial tree.

Informally, a (k -dimensional) spatial tree is a pair (\mathcal{T}, V) where $\mathcal{T} \in \mathbb{T}$ and V is a continuous mapping from \mathcal{T} into \mathbb{R}^k . Since we defined \mathbb{T} as a space of equivalence classes of trees, we should be a little more precise at this point. If \mathcal{T} and \mathcal{T}' are two (rooted compact) real trees and V and V' are \mathbb{R}^k -valued continuous mappings defined respectively on \mathcal{T} and \mathcal{T}' , we say that the pairs (\mathcal{T}, V) and (\mathcal{T}', V') are equivalent if there exists a root-preserving isometry Φ from \mathcal{T} onto \mathcal{T}' such that $V'_{\Phi(\sigma)} = V_\sigma$ for every $\sigma \in \mathcal{T}$. A spatial tree is then defined as an equivalent class for the preceding equivalence relation, and we denote by \mathbb{T}_{sp} the space of all spatial trees. Needless to say we will often abuse notation and identify a spatial tree with an element of the corresponding equivalent class.

We denote by \mathbb{T}_{sp} the set of all spatial trees. Recall the notation of Section 1. We define a distance on \mathbb{T}_{sp} by setting

$$d_{sp}((T, V), (T', V')) = \frac{1}{2} \inf_{\mathcal{R} \in \mathcal{C}(T, T'), (\rho, \rho') \in \mathcal{R}} \left(\text{dis}(\mathcal{R}) + \sup_{(\sigma, \sigma') \in \mathcal{R}} |V_\sigma - V'_{\sigma'}| \right),$$

where ρ and ρ' obviously denote the respective roots of T and T' . It is easy to verify that $(\mathbb{T}_{sp}, d_{sp})$ is a Polish space.

Let us fix $x \in \mathbb{R}^k$. Also let $T \in \mathbb{T}$ be a compact rooted real tree with root ρ and metric d . We may consider the \mathbb{R}^k -valued Gaussian process $(Y_\sigma, \sigma \in T)$ whose distribution is characterized by

$$\begin{aligned} E[Y_\sigma] &= x, \\ \text{cov}(Y_\sigma, Y_{\sigma'}) &= d(\rho, \sigma \wedge \sigma') \text{Id}, \end{aligned}$$

where Id denotes the k -dimensional identity matrix. The formula for the covariance is easy to understand if we recall that $\sigma \wedge \sigma'$ is the most recent common ancestor to σ and σ' , and so the ancestors of σ and σ' are the same up to level $d(\emptyset, \sigma \wedge \sigma')$. Note that

$$\text{cov}(Y_\sigma - Y_{\sigma'}, Y_\sigma - Y_{\sigma'}) = d(\sigma, \sigma') \text{Id}.$$

Let $\mathcal{N}(T, \delta)$ stand for the minimal number of balls with radius δ needed to cover T . From Theorem 11.17 in [41], we know that under the condition

$$\int_0^1 (\log \mathcal{N}(T, \varepsilon^2))^{1/2} d\varepsilon < \infty, \quad (8)$$

the process $(Y_\sigma, \sigma \in T)$ has a continuous modification. We keep the notation Y for this modification. Assuming that (8) holds, we denote by Q_T^x the law on \mathbb{T}_{sp} of $(T, (Y_\sigma, \sigma \in T))$.

It is easy to verify that condition (8) holds $\Theta_\alpha(dT)$ a.e., for every $\alpha \in (1, 2]$. The definition of Q_T^x then makes sense $\Theta_\alpha(dT)$ a.e., and we may set

$$\mathbb{N}_x^\alpha = \int \Theta_\alpha(dT) Q_T^x,$$

which defines a σ -finite measure on \mathbb{T}_{sp} .

We can now turn to connections with superprocesses. Under the measure \mathbb{N}_x^α , we may for every $a > 0$ define a measure $\mathcal{Z}_a = \mathcal{Z}_a(T, V)$ on \mathbb{R}^k by setting

$$\langle \mathcal{Z}_a, \varphi \rangle = \int \ell^a(d\sigma) \varphi(V_\sigma),$$

where the local time measure ℓ^a was introduced in Theorem 5.1. The next proposition reformulates a special case of Theorem 4.2.1 in [22].

Theorem 6.1 *Let $\gamma \in M_f(\mathbb{R}^k)$ and let*

$$\sum_{i \in \mathcal{I}} \delta_{(\mathcal{T}^i, V^i)}$$

be a Poisson point measure on \mathbb{T}_{sp} with intensity $\int \gamma(dx) \mathbb{N}_x^\alpha$. Then the process $(Z_a, a \geq 0)$ defined by

$$\begin{aligned} Z_0 &= \gamma, \\ Z_a &= \sum_{i \in \mathcal{I}} \mathcal{Z}_a(\mathcal{T}^i, V^i), \quad a > 0, \end{aligned}$$

is a super-Brownian motion with branching mechanism $\psi(u) = u^\alpha$ ($\psi(u) = 2u^2$ if $\alpha = 2$) started at γ .

In the formula for Z_a , only finitely many terms can be nonzero, simply because finitely many trees in the collection $(\mathcal{T}^i, i \in \mathcal{I})$ are such that $h(\mathcal{T}^i) > a$. From the continuity properties of local times, we see that the version of Z defined in the proposition is càdlàg on $(0, \infty)$ for the weak topology on finite measures on \mathbb{R}^k . By the known regularity properties of superprocesses (see e.g. the more general Theorem 2.1.3 in [15]), it must indeed be càdlàg on $[0, \infty)$.

Theorem 6.1 is clearly related to the Brownian snake construction of superprocesses (in the case $\alpha = 2$) or more generally to the Lévy snake of [48] or [21]. Consider the case $\alpha = 2$ and assume that the spatial displacements have constructed (with initial point x) using the tree \mathcal{T}_e (where e is distributed according to the Itô measure). For every $s \geq 0$, we may consider the path consisting of the spatial positions V_σ along the ancestor line of the vertex $p_e(s)$. This gives a random path W_s with duration $d_e(\rho, p_e(s)) = e(s)$. The process $(W_s)_{s \geq 0}$ is then the path-valued Markov process called the Brownian snake, here constructed under its excursion measure from x .

In view of Theorem 6.1, the measures \mathbb{N}_x^α (or rather the distribution under \mathbb{N}_x^α of the measure-valued process $(Z_a, a \geq 0)$) are called the excursion measures of the ψ -super-Brownian motion. In the quadratic branching case, these measures play an important role in the study of connections between superprocesses and partial differential equations: See in particular [43]. In the case of a general branching mechanism, excursion measures are constructed via the Lévy snake in Chapter 4 of [22], and a different approach has been proposed recently by Dynkin and Kuznetsov [26].

Connections with partial differential equations are helpful to analyse the probabilistic properties of spatial trees. We content ourselves with one statement concerning the range

$$\mathcal{R} = \{V_\sigma : \sigma \in \mathcal{T}\}.$$

Theorem 6.2 *Let K be a compact subset of \mathbb{R}^k . Then the function*

$$u(x) = \mathbb{N}_x^\alpha(\mathcal{R} \cap K \neq \emptyset), \quad x \in \mathbb{R}^k \setminus K$$

is the maximal nonnegative solution of $\frac{1}{2}\Delta u = u^\alpha$ ($\frac{1}{2}\Delta u = 2u^2$ if $\alpha = 2$) in $\mathbb{R}^k \setminus K$.

This theorem is a reformulation in our formalism of results proved by Dynkin [25] in the framework of the theory of superprocesses.

Integrated super-Brownian excursion (ISE). Let us return for a while to the CRT, and combine the branching structure of the CRT with k -dimensional Brownian motions started from $x = 0$, in the way explained in the previous section. Precisely this means that we are considering the probability measure on \mathbb{T}_{sp} defined by

$$\mathbb{N}_0^{2,(1)} = \int \Theta_2^{(1)}(d\mathcal{T}) Q_{\mathcal{T}}^0,$$

where $\Theta_2^{(1)}(d\mathcal{T})$ is the law of the CRT, in agreement with our previous notation $\Theta_\alpha^{(1)}$. Recall the notation \mathbf{m} for the uniform measure on \mathcal{T} (this makes sense $\Theta_2^{(1)}(d\mathcal{T})$ a.s.).

Definition 6.1 *The random probability measure U on \mathbb{R}^k defined under $\mathbb{N}_0^{2,(1)}$ by*

$$\langle U, \varphi \rangle = \int \mathbf{m}(d\sigma) \varphi(V_\sigma)$$

is called k -dimensional ISE (for Integrated Super-Brownian Excursion).

Note that the topological support of ISE is the range \mathcal{R} of the spatial tree, and that ISE should be interpreted as the uniform measure on this set. The random measure ISE was first discussed by Aldous [4]. It occurs in various asymptotics for models of statistical mechanics: See in particular [17] and [34].

7 Conditioned spatial trees and quadrangulations

In this section, we study a conditioning problem for the spatial trees of the previous section, and we then explain why this conditioning problem is related to certain asymptotics for planar quadrangulations. We consider **one-dimensional** spatial displacements ($k = 1$) and we write $\mathbb{N}_0^{(1)}$ for $\mathbb{N}_0^{2,(1)}$ since we will only consider the case $\alpha = 2$. Recall the notation $(\mathcal{T}, (V_\sigma)_{\sigma \in \mathcal{T}})$ for the generic element of \mathbb{T}_{sp} , and \mathcal{R} for the associated range.

Our goal is to define the probability measure $\mathbb{N}_0^{(1)}$ conditioned on the event $\{\mathcal{R} \subset [0, \infty)\}$. In other words we consider a tree of Brownian paths started from the origin, whose underlying genealogical structure is the CRT, and we want to condition this tree to remain on the positive half-line. Obviously this conditioning is very degenerate (it is already degenerate if one considers a single Brownian path started from the origin).

Theorem 7.1 ([50]) *We have*

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{-4} \mathbb{N}_0^{(1)}(\mathcal{R} \subset (-\varepsilon, \infty)) = \frac{2}{21}.$$

There exists a probability measure on \mathbb{T}_{sp} , which is denoted by $\overline{\mathbb{N}}_0^{(1)}$, such that

$$\lim_{\varepsilon \downarrow 0} \mathbb{N}_0^{(1)}(\cdot \mid \mathcal{R} \subset (-\varepsilon, \infty)) = \overline{\mathbb{N}}_0^{(1)},$$

in the sense of weak convergence in the space of probability measures on \mathbb{T}_{sp} .

The measure $\overline{\mathbb{N}}_0^{(1)}$ is the law of the conditioned spatial tree we were aiming at. The next result will show that, rather remarkably, this conditioned tree can be obtained from the unconditioned one simply by re-rooting it at the vertex corresponding to the minimal spatial position.

Before stating this result we need some notation for re-rooted trees. If (\mathcal{T}, V) is a spatial tree, and $\sigma_0 \in \mathcal{T}$, the re-rooted spatial tree $(\mathcal{T}^{[\sigma_0]}, V^{[\sigma_0]})$ is defined by saying that $\mathcal{T}^{[\sigma_0]}$ is \mathcal{T} re-rooted at σ_0 , and $V_\sigma^{[\sigma_0]} = V_\sigma - V_{\sigma_0}$ for every $\sigma \in \mathcal{T}$.

Theorem 7.2 ([50]) *There is $\mathbb{N}_0^{(1)}$ a.s. a unique vertex σ_* which minimizes V_σ over $\sigma \in \mathcal{T}$. The probability measure $\overline{\mathbb{N}}_0^{(1)}$ is the law under $\mathbb{N}_0^{(1)}$ of the re-rooted spatial tree $(\mathcal{T}^{[\sigma_*]}, V^{[\sigma_*]})$.*

This theorem is reminiscent of a famous theorem of Verwaat [60] connecting the Brownian bridge and the normalized Brownian excursion. The proof of Theorem 7.2 makes a heavy use of the invariance property under re-rooting (Proposition 5.3).

It is interesting to reinterpret the preceding theorem in terms of ISE. If we want to define one-dimensional ISE conditioned to put no mass on the negative half-line, a natural way is to condition it to put no mass on $] - \infty, -\varepsilon[$ and then to let ε go to 0. As a consequence of the previous two theorems, this is equivalent to shifting the unconditioned ISE to the right, so that the left-most point of its support becomes the origin.

Our motivation for studying conditioned spatial trees came from applications to planar quadrangulations. In order to describe these applications, we first need to introduce well-labelled trees. We call labelled tree any pair (θ, v) where $\theta \in \mathbb{T}$ is a plane tree (cf Section 3) and v is a mapping from θ into the set \mathbb{Z} of integers. This is the obvious discrete analogue of the spatial trees considered above. We say that (θ, v) is a well-labelled tree if in addition $v(\varnothing) = 1$, $v(u) \geq 1$ for every $u \in \theta$ and $|v(u) - v(u')| \leq 1$ whenever u and u' are neighboring vertices (that is, u is the father of u' or u' is the father of u). Again, this is a discrete version of the conditioned spatial trees discussed in the preceding theorems.

In a way similar to what we did in Section 3, there is an obvious way of viewing a labelled tree as a spatial tree: If (θ, v) is a labelled tree, let \mathcal{T}^θ be as in Section 3, and define $V^{(\theta, v)}$ by setting $V^{(\theta, v)}(\sigma) = v(u)$ if σ is the vertex of \mathcal{T}^θ corresponding to u , and then interpolating linearly between neighboring vertices to complete the definition of $V^{(\theta, v)}$.

Let \mathcal{W}_n stand for the set of all well-labelled trees with n edges (or equivalently $n + 1$ vertices).

Theorem 7.3 *The law of the rescaled tree $((2n)^{-1/2}\mathcal{T}^\theta, (9/8)^{1/4}n^{-1/4}V^{(\theta,v)})$ under the uniform probability measure on \mathcal{W}_n converges as $n \rightarrow \infty$ towards the measure $\overline{\mathbb{N}}_0^{(1)}$.*

This theorem is a consequence of more general statements obtained in [46] for conditioned Galton-Watson trees. Closely related results can be found in [13] and [52]. The factor $(9/8)^{1/4}$ is easy to understand if we write $(9/8)^{1/4} = 2^{-1/4}(2/3)^{-1/2}$ and note that $2/3$ is the variance of the uniform distribution on $\{-1, 0, 1\}$ (while the factor $2^{-1/4}$ corresponds to the term $2^{-1/2}$ in $(2n)^{-1/2}\mathcal{T}^\theta$).

Remark Without the positivity constraint (for instance, considering labelled trees with the only properties that the label of the root is 0 and the labels of two neighboring vertices differ by at most 1), the limiting distribution would be $\mathbb{N}_0^{(1)}$ instead of $\overline{\mathbb{N}}_0^{(1)}$. This unconditional analogue of Theorem 7.3 follows rather easily from Theorem 3.1: See Janson and Marckert [39] for much more general statements of this type (Kesten [38] discusses related results for tree-indexed random walks under different conditionings of the underlying trees).

Let us now discuss quadrangulations. A planar quadrangulation is a planar map where each face, including the unbounded one, has degree 4. We are interested in rooted quadrangulations, meaning that we distinguish an edge on the border of the infinite face, which is oriented counterclockwise and called the root edge. The origin of the root edge is called the root vertex. Two rooted quadrangulations are considered identical if there is a homeomorphism of the plane that sends one map onto the other. We refer to Chassaing and Schaeffer [13] for more precise definitions. We denote by \mathcal{Q}_n the set of all rooted quadrangulations with n faces. A key result gives a bijection between the sets \mathcal{W}_n and \mathcal{Q}_n (see Theorem 1 in [13], and note that this bijection has been extended recently to more general planar maps by Bouttier, Di Francesco and Guitten [11]). Moreover the radius of the quadrangulation, defined as the maximal graph distance between the root vertex and another vertex, corresponds via this bijection to the maximal label of the tree. We can then deduce the following corollary ([13], Corollary 3).

Corollary 7.4 *Let R_n denote the radius of a random quadrangulation chosen uniformly among all rooted planar quadrangulations with n faces. Then,*

$$n^{-1/4}R_n \xrightarrow[n \rightarrow \infty]{(d)} \rho$$

where the limiting variable ρ is distributed as

$$\left(\frac{8}{9}\right)^{1/4} \left(\sup_{\sigma \in \mathcal{T}} V_\sigma - \inf_{\sigma \in \mathcal{T}} V_\sigma \right)$$

under $\mathbb{N}_0^{(1)}$ (alternatively, ρ is distributed as $(8/9)^{1/4}$ times the length of the support of one-dimensional ISE).

The proof of the corollary is easy from the preceding observations. Properties of the bijection between the sets \mathcal{W}_n and \mathcal{Q}_n imply that R_n has the same distribution as $\sup\{v(u) : u \in \theta\}$ under the uniform probability measure on \mathcal{W}_n . By construction,

$$n^{-1/4} \sup\{v(u) : u \in \theta\} = n^{-1/4} \sup\{V^{(\theta,v)}(\sigma) : \sigma \in \mathcal{T}^\theta\}.$$

By Theorem 7.3, the law of the latter quantity under the uniform probability measure on \mathcal{W}_n converges to the law of

$$\left(\frac{8}{9}\right)^{1/4} \sup_{\sigma \in \mathcal{T}} V_\sigma$$

under $\overline{\mathbb{N}}_0^{(1)}$. By Theorem 7.2, this distribution is the same as the limiting one in Corollary 7.4. Note some information about this limiting distribution can be found in Delmas [16] and in the recent preprint [8].

Large random quadrangulations, or more general planar maps, are used in theoretical physics as models of random surfaces (see in particular [9], [10] and [24]). Assuming that there is a limiting continuous object for uniform random quadrangulations with n faces, the normalizing factor $n^{-1/4}$ in Corollary 7.4 suggests that its fractal dimension should be 4, a fact that is widely believed in the physics literature. A recent paper of Marckert and Mokkadem [52] uses ideas related to the previous discussion to construct the Brownian map, which is a candidate for the continuous limit of random quadrangulations.

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